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INTEGRAL ESTIMATION AND ADAPTIVE STABILIZATION OF NON-HOLONOMIC CONTROLLED SYSTEMS†

V. YU. TERTYCHNYI

St Petersburg

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A method for the programmed stabilization of non-holonomic dynamic systems is proposed. The original problem is reduced to a constrained adaptive control problem with unknown perturbations, which are represented by the reactions of linear (not necessarily ideal) non-holonomic constraints. Effective control and parameter estimation algorithms are constructed for the exponential stabilization of the system. The method can be extended to non-holonomic systems whose parameters are not known in advance or undergo an unknown bounded drift with time.

THE STABILIZATION of controlled systems with holonomic constraints [1–4] and the stabilization of their adaptive [5–9] and stochastic [5, 10, 11] versions have been considered previously. It is of interest to consider variable-structure deterministic systems and systems operating in the presence of unknown parameter drift. If the drift model is known, the adaptive feedback design scheme can be reduced to standard recursive estimation procedures. If the parameter drift model is unknown, the control system is chosen using filters [7, 12]. The stabilization of systems with a memory under prior uncertainty using proportional-plus-integral controllers has been considered in [13, 14].

1. STATEMENT OF THE PROBLEM

Consider a non-holonomic system whose dynamics are described in generalized coordinates by equations with undetermined Lagrange multipliers

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = u_i + \sum_{k=1}^{r} \lambda_k \frac{\partial f_k}{\partial q_i^{\cdot}} \tag{1.1}$$

$$T = \frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(q) q_i^{\cdot} q_j^{\cdot}, \quad q(0) = q_0, \quad q^{\cdot}(0) = q_0^{\cdot} \tag{1.2}$$

where $q(t)$, $u(t) \in R^n$ are vectors of generalized coordinates and generalized controls, respectively, and λ_k are the undetermined Lagrange multipliers in the expression for the constraint reaction forces. We will assume that the vectors $q(t)$ and $q^{\cdot}(t)$ are observable in the control system at any time; T is the kinetic energy and A_{ij} are the elements of the symmetric positive definite square matrix $A(q)$. The system is subjected to r non-holonomic first-order constraints linear in $q_i^{\cdot}$, which, in general, are represented by smooth hypersurfaces of the form

$$f_k(q_i, q_i^{\cdot}, t) = 0 \quad (k = 1, 2, \ldots, r) \tag{1.3}$$

subject to the constraints

$$\sum_{i=1}^{n} \frac{\partial f_k}{\partial q_i^{\cdot}} \delta q_i = 0, \quad \operatorname{rank}\left\| \frac{\partial f_k}{\partial q_i^{\cdot}} \right\| = r \tag{1.4}$$

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In (1.4), the first equality is Chetayev's condition [15] imposed on the allowed virtual displacements δq_i (for linear non-holonomic constraints this condition is obviously always satisfied), and the second equality is the condition for the constraints (1.3) to be independent, which enables r dependent velocities to be expressed in terms of the s independent velocities ($s = n - r$ is the number of degrees of freedom) [16]

$$f_k(q_i, \dot{q}_i, t) = \dot{q}_{s+k} - \varphi_k(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, t) = 0 \tag{1.5}$$

Henceforth, we shall use the notation $q_\alpha(t)$ $[\dot{q}_\alpha(t)] \in R^s$ for the independent coordinates (velocities) and $q_\beta(t)$ $[\dot{q}_\beta(t)] \in R^r$ for the dependent coordinates (velocities). In this notation, $q(t) = [q_\alpha(t), q_\beta(t)]$. Then (1.5) may be rewritten in vector form as

$$\dot{q}_\beta = \dot{q}_\beta(q, \dot{q}_\alpha, t) \tag{1.6}$$

Substituting (1.2) into (1.1), we obtain the vector-matrix form of the equations of motion

$$A(q)\ddot{q} + B(q, \dot{q}) = u + D(q)\lambda \tag{1.7}$$

where $\lambda(t) \in R^r$ is the vector of undetermined Lagrange multipliers, and $D(q)$ is the $n \times r$ structure matrix with the elements $\partial f_k / \partial \dot{q}_i$ ($k = 1, 2, \ldots, r$, $i = 1, 2, \ldots, n$). We may assume without loss of generality that the constraint equations are time-independent. We now use (1.6) to substitute the expressions for $\ddot{q}_\beta$ into Eq. (1.7)

$$A(q) \left\| \begin{matrix} \ddot{q}_\alpha \\ F_1(q, \dot{q})\dot{q} + F_2(q)\ddot{q}_\alpha \end{matrix} \right\| + B(q, \dot{q}) = u + D(q)\lambda$$

$F_1(q, \dot{q})$ is the $r \times n$ matrix with elements $\partial \dot{q}_\beta / \partial q_i$, and $F_2(q)$ is the $r \times s$ matrix with elements $\partial \dot{q}_\beta / \partial \dot{q}_\alpha$, $s = n - r$. Separating the vector of independent accelerations $\ddot{q}_\alpha$ in this equation, we obtain

$$\begin{aligned} &S_1(q)\ddot{q}_\alpha + S_2(q, \dot{q}) = u + D(q)\lambda \\ &S_1(q) = A_1(q) + A_2(q)F_2(q) \\ &S_2(q, \dot{q}) = B(q, \dot{q}) + A_2(q)F_1(q, \dot{q})\dot{q} \end{aligned} \tag{1.8}$$

where $S_1(q)$ is an $n \times s$ matrix, $S_2(q, \dot{q})$ is an $n \times 1$ vector, $A(q) = \|A_1(q) | A_2(q)\|$, $A_1(q)$ is an $n \times s$ matrix, and $A_2(q)$ is an $n \times r$ matrix.

A specific feature of Eqs (1.8) is that they contain $n \times r$ unknowns. Since r equations of the constraints have been used to separate the independent accelerations, we cannot use Eqs (1.3) and (1.6) again to find all the unknowns. Equation (1.8) thus includes the unknown deterministic perturbations $v(q, t) = D(q)\lambda(t)$, where $\lambda(t)$ plays the role of an unknown drifting vector parameters, and the control system itself becomes adaptive. We will assume that the perturbations $v(t)$ are uniformly bounded ($\sup_{0 \leqslant t \leqslant t_1} \|v(t)\| < C_v$) and that the bound C_v on the Euclidean norm in a finite time interval is also unknown.

In (1.1) it is required to construct:

1. an adaptive feedback control as a function of the observable arguments

$$u = u(q(t), \dot{q}(t), q_{\alpha p}(\cdot), v_*(t)), \quad \|u\| < C_u$$

where C_u is a given positive constant, $v_*(t) \in R^n$ is the estimated parameter vector, and $q_{\alpha p}(t)$ is the programmed motion of the system in the independent coordinates defined in the entire finite time interval $[0, t_1]$ (the time t_1 is not fixed);

2. an algorithm for estimating $v_*(t)$,

$$\dot{v}_* = \dot{v}_*(q(t), \dot{q}(t), q_{\alpha p}(\cdot), v_*(t))$$

which ultimately satisfies the objective conditions

$$\|q_\alpha(t) - q_{\alpha p}(t)\| < \delta_1, \quad \|v_*(t) - v(t)\| < \delta_2 \tag{1.9}$$

where δ_1, $\delta_2 > 0$ is the given stabilization accuracy.

The second inequality in (1.9) obviously ensures tracking of the system constraints with a certain accuracy $\delta_3(\delta_2)$; the estimates v_* and λ_* are related by the algebraic equalities

$$v_* = D\lambda_*, \quad \lambda_* = D^+ v_*, \quad D^+ = (D'D)^{-1}D', \quad \operatorname{rank} D = r$$

where the prime denotes the transpose.

2. ADAPTIVE FILTERING

In Eq. (1.8), we assume the following control law

$$\begin{aligned} u &= S_2(q, q^{\cdot}) + S_1(q)q_* - v_* \\ q_* &= q^{\cdot\cdot}_{\alpha p} - \beta_1(q^{\cdot}_{\alpha} - q^{\cdot}_{\alpha p}) - \beta_2(q_{\alpha} - q_{\alpha p}) \end{aligned} \tag{2.1}$$

where β_1, $\beta_2 > 0$ are given numbers. As the estimation algorithm for choosing v_*, we take the first-order differential equation

$$v^{\cdot}_* - v^{\cdot} + \gamma(v_* - v) + v^{\cdot} e^{-\gamma t} = 0 \tag{2.2}$$

($\gamma > 0$ is a given number) with the solution

$$v_*(t) = (1 - e^{-\gamma t})v(t) + e^{-\gamma t} v_*(0)$$

that ensures satisfaction of the second objective condition in (1.9) in a finite time interval. Substituting (2.1) into the equation of motion (1.8), we obtain

$$S_1(q)(q^{\cdot\cdot}_{\alpha} - q_*) = v - v_* \tag{2.3}$$

Noting that $S_1(q)$ is bounded and the norm $\|v - v_*\|$ is exponentially decreasing, we obtain the equations

$$q^{\cdot\cdot}_{\alpha i} - q^{\cdot\cdot}_{\alpha p i} + \beta_1(q^{\cdot}_{\alpha i} - q^{\cdot}_{\alpha p i}) + \beta_2(q_{\alpha i} - q_{\alpha p i}) = \psi_i \quad (i = 1, 2, \ldots, s)$$

which obviously ensures satisfaction of the first objective condition in (1.9) when the scalar function $\psi_i(t)$ tends exponentially to zero.

Let us extend the definition of the estimation algorithm (2.2). To this end, we solve Eq. (2.3) for the vector v

$$v = v_* + G, \qquad G = S_1(q^{\cdot\cdot}_{\alpha} - q_*) \tag{2.4}$$

and substitute v, $v^{\cdot}$ from (2.4) into the convergent algorithm (2.2) to obtain estimates of v_*

$$(e^{-\gamma t} - 1)G^{\cdot} - \gamma G + e^{-\gamma t} v^{\cdot}_* = 0. \tag{2.5}$$

Note that the coefficients in Eq. (2.5) depend on q, $q^{\cdot}$, $q^{\cdot\cdot}$, $q^{\cdot\cdot\cdot}$, whereas the control systems are limited to measuring only q and $q^{\cdot}$. We accordingly integrate Eq. (2.4) twice with weight $e^{-\kappa(t-s)}$ ($\kappa > 0$) over the interval $[0, t]$.

Let

$$\langle v \rangle_{-\kappa} = \int_0^t e^{-\kappa(t-s)} v(s)ds, \quad [v]_{-\kappa} = \int_0^t \int_0^s e^{-\kappa(t-r)} v(r)drds$$

We replace the estimation altorithm (2.2) with the smoothed analogue

$$V^{\cdot}_* - V^{\cdot} + \gamma(V_* - V) + V^{\cdot} e^{-\gamma t} = 0 \tag{2.6}$$

where $V(t)$ and $V_*(t)$ are the outputs of the filter

$$V^{\cdot} + \kappa V = v(\kappa - \xi)e^{-\xi t} + \langle v \rangle_{-\kappa} \tag{2.7}$$

with the solution

$$V(t) = \nu e^{-\xi t} + [v]_{-\kappa}, \qquad V(0) = \nu$$

$$V^{\cdot}(t) = -\nu\xi e^{-\xi t} + \langle v\rangle_{-\kappa} - \kappa[v]_{-\kappa}$$

and the filter

$$V_*^{\cdot} + \kappa V_* = \nu_*(\kappa - \xi)e^{-\xi t} + \langle v_*\rangle_{-\kappa} \tag{2.8}$$

with the corresponding solution, where $V_*(0) = \nu_*$; Φ, $\kappa>0$, ν, ν_* are given numbers and vectors. Substituting the solutions of the filters (2.7) and (2.8) into (2.6), we obtain the estimation algorithm in the following form

$$\begin{aligned}\langle v_* - v\rangle_{-\kappa} &= (\nu_* - \nu)(\xi - \gamma)e^{-\xi t} - (\gamma - \kappa)[v_* - v]_{-\kappa} + \nu\xi e^{-(\xi+\gamma)t} - \\ &- e^{-\gamma t}\langle v\rangle_{-\kappa} + \kappa e^{-\gamma t}[v]_{-\kappa}\end{aligned} \tag{2.9}$$

We will show that algorithm (2.9) converges for $\sigma = \gamma - \kappa > 0$, $\tau = \xi - \kappa > 0$. Using the notation

$$\Delta(t) = v_*(t) - v(t), \quad x(t) = \Delta(t)e^{\kappa t}, \quad y(t) = \int_0^t x(s)ds, \quad F(t) = \int_0^t y(s)ds,$$

$$P(t) = (\nu_* - \nu)(\xi - \gamma)e^{-\xi t} + \nu\xi e^{-(\xi+\gamma)t} - e^{-\gamma t}\langle v\rangle_{-\kappa} + \kappa e^{-\gamma t}[v]_{-\kappa}$$

we obtain the solution of Eq. (2.9) $F^{\cdot}(t) = -\sigma F(t) + e^{\kappa t}P(t)$ in the form

$$F(t) = \int_0^t y(s)ds = \nu_*(e^{-\sigma t} - e^{-\tau t}) + \nu e^{-\tau t}(1 - e^{-\gamma t}) - e^{-\sigma t}[v]_{-\kappa} + \kappa e^{-\sigma t}\int_0^t [v]_{-\kappa}ds$$

Differentiating twice with respect to t and multiplying by $e^{-\kappa t}$, we establish an exponential decrease of the norm $\|v_*(t) - v(t)\|$ as t increases which in turn ensures that the first objective condition in (1.9) is satisfied. If we now integrate Eq. (2.4) twice with weight $e^{-\kappa(t-s)}$ and substitute the result into (2.9), we obtain a convergent algorithm for estimating v_*

$$\begin{aligned}\langle(e^{-\gamma t} - 1)G\rangle_{-\kappa} - (\sigma - \kappa e^{-\gamma t})[G]_{-\kappa} &= (\nu_* - \nu)(\xi - \gamma)e^{-\xi t} + \\ + \nu\xi e^{-(\xi+\gamma)t} - e^{-\gamma t}\langle v_*\rangle_{-\kappa} &+ \kappa e^{-\gamma t}[v_*]_{-\kappa}\end{aligned} \tag{2.10}$$

which ensures that the objective inequalities are satisfied in a finite time interval.

3. RECURSIVE ESTIMATION

Let us convert Eq. (2.10) into a form such that the algorithm depends only on q and $q^{\cdot}$. To this end, we will naturally use the fact that the vector $q_\alpha^{\cdot\cdot}$ occurs linearly in G. Omitting the elementary algebra associated with integration by parts, we obtain in the new notation

$$\langle v_*\rangle_{-\kappa} - \kappa[v_*]_{-\kappa} = \Phi(t)e^{-\kappa t} \tag{3.1}$$

$$\Phi(t) = \Phi_1(t) + \Phi_2(t), \quad \Phi_1(t) = (e^{\gamma t} - 1)(\varphi(t) - a - \int_0^t \psi(s)ds) +$$

$$+ (\sigma e^{\gamma t} - \kappa)\int_0^t(\varphi(s) - a - \int_0^s \psi(r)dr)ds, \quad \varphi(t) = S_1(t)q_\alpha^{\cdot}(t)e^{\kappa t},$$

$$\psi(t) = \kappa\varphi(t) + (S_1(t)q_\alpha^{\cdot}(t) - S_1(t)q_*(t))e^{\kappa t}, \quad a = S_1(0)q_\alpha^{\cdot}(0),$$

$$\Phi_2(t) = (\nu_* - \nu)(\xi - \gamma)e^{(\gamma-\tau)t} + \nu\xi e^{-\tau t}$$

Solving the differential equation (3.1) for $e^{\kappa t}[v_*]_{-\kappa}$, we obtain a Volterra linear integral equation of the first kind

$$\begin{aligned}\langle v_*\rangle_{-\kappa} &= \Psi(t) \\ \Psi(t) &= (\kappa\langle\Phi\rangle_\kappa + \Phi(t))e^{-\kappa t}\end{aligned} \tag{3.2}$$

A specific feature of Eq. (3.2) is that it cannot be differentiated: the right-hand side $\Psi(t)$ depends on q and $q^{\bullet}$, and we must obviously use some approximate method [17, 18]: the method of moments, the kernel change method, the polynomial method, the coefficient averaging method, the oscillating function method, etc. The main condition for uniform convergence of recursive algorithms in the interval of definition is the continuous differentiability of the kernels with respect to all arguments. If we restrict the analysis to approximation by a numerical quadrature formula with the integration nodes $t_0, t_1, \ldots, t_N$, the recursive system of equations for finding the estimates v_* takes the form

$$\sum_{k=0}^{N} C_k e^{\kappa t_k} v_*(t_k) = \Psi(t_N) e^{\kappa t_N} \tag{3.3}$$

where C_k are the coefficients of the corresponding quadrature formula. The system of equations (3.3) provides the most efficient method of replacing the integral equation (3.2) [17, 18].

4. THE CHAPLYGIN–CARATHÉODORY PROBLEM

As an application of the proposed method for choosing an adaptive control system, consider the model non-holonomic motion of a rigid body in a stationary horizontal plane Ω. The body has three points of contact with the plane, two of which slide without friction on the plane and the third is the point of contact of a sharp runner (a blade) rigidly connected to the body [19]. The point of contact of the blade K can move freely on the plane Ω along the blade, but not perpendicularly to the blade.

We introduce two systems of coordinates (Fig. 1): the fixed system Oxy and the moving system Kx_1y_1 attached to the body. The Kx_1 axis is directed along the runner and the Ky_1 axis is perpendicular to Kx_1. The position of the body (a trolley) is defined by three generalized coordinates: the coordinates x, y of the point K and the angle φ between the axes Ox and Kx_1. Let m be the mass of the body, a and b the coordinates of the centre of mass C in the system Kx_1y_1, and J_0 the moment of inertia of the trolley about the vertical through the point C. We will assume that three generalized control forces act along the three generalized coordinates. The constraint imposed on the system ensures that the velocity of the point K is always directed along the Kx_1 axis, i.e. $y^{\bullet} = x^{\bullet}\,\mathrm{tg}\,\varphi$. The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}(mv_C^2 + J_0\dot{\varphi}^2) = \\ &= \tfrac{1}{2}m\{[\dot{x} - \dot{\varphi}(a\sin\varphi + b\cos\varphi)]^2 + [\dot{y} + \dot{\varphi}(a\cos\varphi - b\sin\varphi)]^2 + k^2\dot{\varphi}^2\} \end{aligned}$$

where k is the radius of the body about the axis through the point C perpendicular to the plane Ω.

The system has two degrees of freedom. As the independent coordinates we take $q_1 = x$ and $q_2 = zv$; the dependent coordinate is $q_3 = y$. We form Eqs (1.1) of controlled motion in each coordinate with one undetermined Lagrange multiplier λ. Then

$$\begin{aligned} & m\ddot{q}_1 + \alpha\ddot{q}_2 - \beta\dot{q}_2^2 = u_1 + \lambda\,\mathrm{tg}\,q_2 \\ & \alpha\ddot{q}_1 + \gamma\ddot{q}_2 + \beta(\ddot{q}_3 - 2\dot{q}_1\dot{q}_2) = u_2 \\ & m\ddot{q}_3 + \beta\ddot{q}_2 + \alpha\dot{q}_2^2 = u_3 - \lambda \\ & \alpha = -m(a\sin q_2 + b\cos q_2), \quad \beta = m(a\cos q_2 - b\sin q_2) \\ & \gamma = m(a^2 + b^2 + k^2) \end{aligned}$$

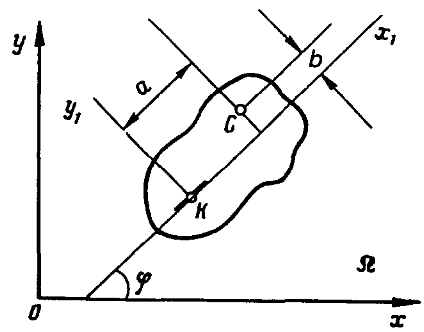

FIG. 1.

The vector-matrix equations (1.7) and (1.8) contain the following matrices and vectors

$$A(q) = \begin{Vmatrix} m & \alpha & 0 \\ \alpha & \gamma & \beta \\ 0 & \beta & m \end{Vmatrix}, \quad B(q, \dot q) = \begin{Vmatrix} -\beta \dot q_2^2 \\ -2\beta \dot q_1 q_2 \\ \alpha \dot q_2^2 \end{Vmatrix}, \quad u = \begin{Vmatrix} u_1 \\ u_2 \\ u_3 \end{Vmatrix}, \quad q = \begin{Vmatrix} q_1 \\ q_2 \\ q_3 \end{Vmatrix}$$

$$q_\alpha = \begin{Vmatrix} q_1 \\ q_2 \end{Vmatrix}, \quad D(q) = \begin{Vmatrix} \mathrm{tg}\, q_2 \\ 0 \\ -1 \end{Vmatrix}, \quad F_1(q, \dot q) = \begin{Vmatrix} 0 & \dfrac{q_1}{\cos^2 q_2} & 0 \end{Vmatrix}, \quad F_2(q) = \begin{Vmatrix} \mathrm{tg}\, q_2 & 0 \end{Vmatrix}$$

$$S_1(q) = \begin{Vmatrix} m & \alpha \\ -mb/\cos q_2 & \gamma \\ m\,\mathrm{tg}\, q_2 & \beta \end{Vmatrix}, \quad S_2(q, \dot q) = \begin{Vmatrix} -\beta \dot q_2^2 \\ \beta \dot q_1 \dot q_2 (1/\cos^2 q_2 - 2) \\ m \dot q_1 \dot q_2/\cos^2 q_2 + \alpha \dot q_2 \end{Vmatrix}$$

In particular, for the simplest case when the projection of the centre of mass is at the point K (the Carathéodory case), we obtain for the control law

$$u = \begin{Vmatrix} 0 \\ 0 \\ m \dot q_1 \dot q_2/\cos^2 q_2 \end{Vmatrix} + \begin{Vmatrix} m & 0 \\ 0 & mk^2 \\ m\,\mathrm{tg}\, q & 0 \end{Vmatrix} \begin{Vmatrix} q_{1*} \\ q_{2*} \end{Vmatrix} + \begin{Vmatrix} v_{1*} \\ v_{2*} \\ v_{3*} \end{Vmatrix}$$

where the estimates v_* are solutions of the recursive linear system of algebraic equations (3.3). When the control u is substituted into the equations of motion, they produce exponential stabilization of the system relative to the programmed trajectories.

5. LOCALLY OPTIMAL STABILIZATION

For many dynamical systems it is important to ensure optimality at any current instant of time. As an example, consider the maximization of the stabilization accuracy over a long time interval. The obvious criterion in this case is a functional which is defined on the current state of the system and is a measure of the perturbed state. Such a functional is provided by an integral quadratic form (a local Lyapunov functional) or by the generalized work criterion [1–5].

To solve the optimal stabilization problem for system (1.7) with the constraints (1.6), we augment the objective conditions (1.9) with the requirement that at each time instant t the following functional is minimized

$$\begin{gathered} J(t) = J[u(t),\ z(t)] = \int_0^t W_1(s)ds + W_2(t) \to \min_{u(t) \in U} \\ W_1 = u' Q_1 u, \ W_2 = z' Q_2 z, \quad z = \dot q_\alpha - \dot q_{\alpha p} + \beta_2 (q_\alpha - q_{\alpha p}) \end{gathered} \tag{5.1}$$

where Q_1 and Q_2 are symmetric matrices that can be factorized in the form $Q_i = X_i' X_i$ $(i = 1, 2)$ for some full-rank rectangular matrix X_i and $Q_2 = S_1' S_1$ is an $s \times s$ matrix.

The optimal control $u_0(t)$ is sought in the class of continuous functions with values in $U \subset R^n$. We also assume that the functional $J(t)$ has a total derivative dJ/dt calculated by Eq. (1.8).

For $W_1(t) = 0$, the functional $J(t) = W_2(t)$ may be treated as the measure of the perturbed state of the process. To find the locally optimal control $u_0(t)$, we use the condition

$$dJ/dt|_{u(t) = u_0(t)} = \min_{u(t)} dJ/dt \tag{5.2}$$

(the necessary and sufficient condition of local optimality).

The objective relationships are obviously satisfied if $z(t)$ tends to zero as t increases, i.e. if the controlled process is asymptotically stable in the measure of the perturbed state $W_2(t)$. If the functionals $W_1(t)$ and $W_2(t)$ are uniformly continuous and positive definite and in (5.2) we have

$$dJ/dt\,|_{u_0(t)} = \min_{u(t)} dJ/dt = 0 \tag{5.3}$$

then the process $z(t)$ $(q_\alpha = q_{\alpha p})$ is asymptotically stable and the control $u_0(t)$ is locally optimal.

Indeed, from (5.2) we directly obtain optimality of the control and an equation for finding the algorithm for estimating $v_*(t)$

$$dJ/dt\,|_{u_0(t)} = [W_1(t) + dW_2(t)/dt]_{u_0(t)} = 0$$

Hence $dW_2(t)/dt < 0$ and the functional $W_2(t)$ satisfies the conditions of Lyapunov's theorem of asymptotic stability.

Differentiating $J(t)$, we obtain by virtue of (1.8) an expression for the optimal control

$$u_0 = -Q_1^{-1} S_1 z. \tag{5.4}$$

If we now substitute (5.4) into dJ/dt, we obtain

$$dJ/dt = z' S_1' Q_1^{-1} S_1 z + d(z' Q_2 z)/dt \tag{5.5}$$

The matrix Q_1^{-1} in (5.5) is defined in the form

$$Q_1^{-1}(v_*) = \operatorname{diag}(v_{1*}^2, v_{2*}^2, \ldots, v_{n*}^2)$$

where v_* are the estimates of the unknown perturbation vector v. Equating (5.5) to zero, we thus ensure that the objective conditions are satisfied.

Cancelling the vector $z'S_1'$, we obtain an equation for the estimates v_*

$$\begin{gathered} 2S_1(q)z^{\cdot} + S_1^{\cdot}(q, q^{\cdot})z + Q_1^{-1}(v_*)S_1(q)z = 0 \\ z^{\cdot} = q_\alpha^{\cdot\cdot} + \varphi, \quad \varphi = -q_{\alpha p}^{\cdot\cdot} + \beta_2(q_\alpha^{\cdot} - q_{\alpha p}^{\cdot}) \end{gathered} \tag{5.6}$$

Let us transform Eq. (5.6) to a form in which its coefficients depend only on q and $q^{\cdot}$

$$\begin{gathered} S_1\varphi - S_2 + \frac{1}{2} S_1^{\cdot} z + v - \frac{1}{2} Q_1^{-1} S_1 z = 0 \\ S_1 q_\alpha^{\cdot\cdot} + S_2 = -Q_1^{-1} S_1 z + v \end{gathered} \tag{5.7}$$

We change to the smoothed analogue of (5.7) by replacing $v(t)$ with the n-dimensional output $V(t)$ of the filter

$$V^{\cdot} + \kappa V = \langle v \rangle_{-\kappa} \tag{5.8}$$

with the solution

$$V(t) = \nu e^{-\kappa t} + [v]_{-\kappa}, \quad V(0) = \nu; \quad \kappa > 0$$

We apply the previous scheme: we integrate twice the equation for v of the system with weight $e^{-\kappa(t-s)}$ over the interval $[0, t]$

$$[v]_{-\kappa} = [\, S_1 q_\alpha^{\cdot\cdot} + S_2 + Q_1^{-1} S_1 z]_{-\kappa} \tag{5.9}$$

Using (5.9), we rewrite the filter equation (5.8) in the form

$$V^{\cdot} + \kappa V = \langle S_2 + Q_1^{-1} S_1 \rangle_{-\kappa} + S_1 q_\alpha^{\cdot} - S_1(0) q_\alpha^{\cdot}(0) e^{-\kappa t} - \langle (\kappa S_1 + S_1^{\cdot}) q_\alpha^{\cdot} \rangle_{-\kappa} \tag{5.10}$$

where the input of the filter (5.10) are the values q, $q^{\cdot}$ and the estimate v_*.

If we now substitute the solution of the filter into the smoothed analogue of Eq. (5.7), we obtain the estimation algorithm in the form

$$\begin{gathered} [Q_1^{-1} S_1 z]_{-\kappa} - \tfrac{1}{2} Q_1^{-1} S_1 z = f(t) \\ f(t) = S_2 - S_1 \varphi - \tfrac{1}{2} S_1^{\cdot} z - \nu e^{-\kappa t} - [S_2]_{-\kappa} - \\ - \langle S_1 q_\alpha^{\cdot} \rangle_{-\kappa} + S_1(0)\, q_\alpha^{\cdot}(0)\, t e^{-\kappa t} + [(\kappa S_1 + S_1^{\cdot}) q_\alpha^{\cdot}]_{-\kappa} \end{gathered} \tag{5.11}$$

where the vector-valued function $f(t)$ depends only on q and $q^{\cdot}$. Equation (5.11) can be represented as a system of integrodifferential equations

$$\int_0^t y(s)\,ds - y^{\cdot}(t) = F(t)$$

$$y(t) = \int_0^t x(s)\,ds, \quad x(t) = Q_1^{-1} S_1 z e^{\kappa t}, \qquad F(t) = f(t) e^{\kappa t}$$

which, as we have shown previously, can be solved numerically by reduction to a recursive system of algebraic linear equations using quadrature formulae.

The adaptive stabilization method proposed in this paper can be extended in a natural way to non-holonomic systems whose internal parameters are not known in advance or undergo an unknown bounded drift with time.

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